Comparison of the Ampère and Biot–Savart magnetostatic force laws in their line-current-element forms

C. Christodoulides
National Technical University of Athens, Department of Physics, Zografou Campus, GR 157 73 Zografou, Athens, Greece

(Received 20 April 1987; accepted for publication 30 June 1987)

The force laws of Ampère and Biot–Savart in magnetostatics are compared using the geometrical model of a closed curve to represent a current loop. The two laws give identical results when forces between separate current loops are considered and also for the force exerted by a current loop on a rectilinear part of itself. According to both laws, these self-forces diverge wherever the curvature of the curve representing the current loop is not equal to zero. Differences in the predictions of the two laws are shown to appear only as differences in diverging forces when evaluating forces of a current loop on a part of itself and to be entirely due to the oversimplified and unrealistic geometrical model used for the current loop.

I. INTRODUCTION

There exist in magnetostatics two laws that give the force between two infinitely thin line-current elements $d\mathbf{l}_1$ and $d\mathbf{l}_2$ through which pass currents I_1 and I_2, respectively. One is the law proposed by Ampère,[1]

$$d^2\mathbf{F}_{2,\mathrm{A}} = -\frac{\mu_0 I_1 I_2}{4\pi}\frac{\mathbf{r}_{12}}{r_{12}^3} \times\left(2(d\mathbf{l}_1\cdot d\mathbf{l}_2) - \frac{3}{r_{12}^2}(d\mathbf{l}_1\cdot\mathbf{r}_{12})(d\mathbf{l}_2\cdot\mathbf{r}_{12})\right), \quad (1)$$

and the other is the law generally known as the Biot–Savart law (also known as Grassmann's equation in its integral form),

$$d^2\mathbf{F}_{2,\mathrm{BS}} = \frac{\mu_0 I_1 I_2}{4\pi}\frac{1}{r_{12}^3}\, d\mathbf{l}_2\times(d\mathbf{l}_1\times\mathbf{r}_{12}), \quad (2)$$

where in both cases the force on element 2 is given and $\mathbf{r}_{12}$ is the position vector of element 2 relative to 1.

The two expressions are different, one of their main differences being the fact that Ampère's law obeys Newton's third law while that of Biot–Savart does not, as can be easily verified by interchanging the subscripts 1 and 2 in the above equations. Nevertheless, it is not possible for isolated current elements to exist in nature, and the two laws should only be compared in their integral forms, referring to complete current loops, bearing in mind that it was experiments on such current loops that led to the formulation of the two laws. In this respect, Maxwell[2] contended that the two laws are indistinguishable, although he did not offer a proof of this, and, at the same time, he showed a preference for the Ampère formulation.

In favor of the Biot–Savart law, it can be said that it has a field-theoretical formulation in terms of the magnetic field of a current loop acting on moving charges in a current element, according to the Lorentz force law. Also it can be derived from the low-velocity approximation of the relativistic expression for the forces between moving charges, i.e, it is based on Coulomb's law, charge invariance, and the Lorentz transformation.[3]

That the two laws predict the same force between separate current loops has been proved,[4] and proof that in this case the Biot–Savart law does not violate Newton's third law can be found in textbooks on electromagnetism.[5] It has also been shown that Eqs. (1) and (2) are special cases of a more general law, consistent with experimental results, which also contains other terms that vanish on integration around closed loops.[6]

There have been, on the other hand, suggestions that experiments designed to measure the force exerted on a part of a current loop by the loop itself may be able to differentiate between the two laws, and experiments have been performed along these lines both in the recent past[7–11] and very recently.[12–14] Differences were reported between measured and theoretically predicted forces, amounting in one case to 40% of the force.[13] No proof was offered that these results differentiate between the two laws as was originally expected.[12,15–18] In fact, more sensitive measurements have recently shown very good agreement between the experimentally measured forces and those predicted by either of the two laws.[19]

At the same time, theoretical proof has been presented of the complete equivalence of the two laws when they are used in the forms involving volume-current distributions, the current elements being volume elements dv with finite current densities $\mathbf{J}$, rather than line-current elements.[20–22] The products $I_1\, d\mathbf{l}_1$ and $I_2\, d\mathbf{l}_2$ in Eqs. (1) and (2) must be replaced by $\mathbf{J}_1\, dv_1$ and $\mathbf{J}_2\, dv_2$, respectively, and the forces must be estimated by volume integration. It was shown that the forces predicted by the two laws as exerted by a complete current distribution on a current element are identical, regardless of whether this is or is not part of the current distribution exerting the force. Consequences of this are that the force on a current element is normal to the current density vector at that point, the distribution of forces and the stresses they produce in conductors are identical according to the two laws,[22] the longitudinal forces along the current density vector claimed to be predicted by Ampère's law[12,17,18,23–27] do not in fact exist,[22,28–30] and also claims that the Biot–Savart law predicts in some cases that a complete current loop exerts a net force on itself thus violating Newton's third law[14,16,17] are without foundation.[21,22] The proof of equivalence of the two laws in magnetostatics was completed by showing that either law could be derived using the other law as a starting point.[20,21]

It has been pointed out that the root of most errors in evaluating the force of a circuit on a part of itself is the use of either law in the form involving infinitely thin line-current elements rather than volume-current elements with finite current density.[21] This practice leads to difficulties with diverging integrals that are bypassed by avoiding the singularities using the wrong regions of integration and obtaining inaccurate results that appear as a disagreement between the two laws.

In this study, it will be taken for granted that the equivalence of the two laws in their volume-current-element forms has been established[20–22] and also that an infinitely thin line-current loop is impossible to exist in nature. As the use of such a geometrical model for a current loop seems to be quite a common practice, however, an attempt will be made to pinpoint the sources of error resulting from its use and, whenever possible, quantify these errors. The expression for the difference of the forces predicted by the two laws will be examined, with special emphasis on the forces exerted by a current loop on a part of itself, their divergence, and the conditions under which these appear to violate Newton's third law. The examination of the question of the divergence of the self-forces in an infinitely thin current loop will be postponed until the end. As a result, the mathematical analysis that will be presented will be more general than will have been necessary, but this will be of assistance in gaining more insight into the possible sources of error when the line-current-element forms of the laws are used.

II. FORCES ON A LINE-CURRENT ELEMENT

Consider a current loop C of infinitesimal thickness, through which flows a steady current I, a shown in Fig. 1. At the origin O of the coordinate system there is a line-current element $I_0\, d\mathbf{l}$. According to the Ampère force law, a line-current element $I\, d\mathbf{r}$ at the position $\mathbf{r}$ on loop C exerts a force on $d\mathbf{l}$ given by

$$d^2\mathbf{F}_A = \frac{\mu_0 I I_0}{4\pi} \frac{\mathbf{r}}{r^3} \left(2(d\mathbf{l}\cdot d\mathbf{r}) - \frac{3}{r^2} (d\mathbf{l}\cdot\mathbf{r})(\mathbf{r}\cdot d\mathbf{r}) \right). \quad (3)$$

For the force on $d\mathbf{l}$, the Biot–Savart law gives

$$d^2\mathbf{F}_{BS} = \frac{\mu_0 I I_0}{4\pi} \frac{1}{r^3} \{ d\mathbf{l} \times [d\mathbf{r} \times (-\mathbf{r})] \} \quad (4)$$

$$= \frac{\mu_0 I I_0}{4\pi} \frac{1}{r^3} [\mathbf{r}(d\mathbf{l}\cdot d\mathbf{r}) - (d\mathbf{l}\cdot\mathbf{r}) d\mathbf{r}]. \quad (5)$$

Subtracting, the difference of the predictions of the two laws for the force on $d\mathbf{l}$ is found to be

$$d^2\mathbf{F} = d^2\mathbf{F}_A - d^2\mathbf{F}_{BS} = \frac{\mu_0 I I_0}{4\pi} \left(\frac{\mathbf{r}}{r^3} (d\mathbf{l}\cdot d\mathbf{r}) + \frac{d\mathbf{r}}{r^3} (d\mathbf{l}\cdot\mathbf{r}) - 3\frac{\mathbf{r}}{r^4} (d\mathbf{l}\cdot\mathbf{r}) dr \right). \quad (6)$$

This may be written as

$$d^2\mathbf{F} = \frac{\mu_0 I I_0}{4\pi} d\left(\frac{\mathbf{r}}{r^3} (d\mathbf{l}\cdot\mathbf{r}) \right), \quad (7)$$

which is an exact differential. An immediate consequence

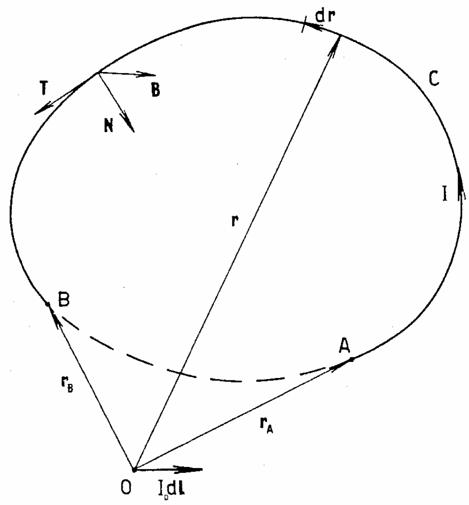

Fig. 1. The geometry of a current loop C and the line-current element $I_0\, d\mathbf{l}$ at the origin. Here, $\mathbf{T}$, $\mathbf{N}$, and $\mathbf{B}$ are the unit tangent, principal normal, and binormal vectors, comprising the moving trihedron of the space curve C.

of this is that integration around a closed space curve C not passing through the origin results in $d\mathbf{F}=0$, or that the Ampère and Biot–Savart forces of a complete current loop on a line-current element, or a part or the whole of another current loop, are equal when these do not have common points with the current loop exerting the force. It should be noted that $\mathbf{r}(d\mathbf{l}\cdot\mathbf{r})/r^3$ is continuous since C is a closed loop. Regarding the force of a complete current loop on a line-current element, it is obvious that since the Biot–Savart force is normal to the current element, so is the Ampère force.

Now the difference of the forces predicted by the two laws as being exerted on $d\mathbf{l}$ by that part of the loop from A to B moving in the direction of the current (continuous line in Fig. 1) is

$$d\mathbf{F}(\mathrm{AB}) = \int_{\mathrm{A}}^{\mathrm{B}} d^2\mathbf{F} = \frac{\mu_0 I I_0}{4\pi}\left[\frac{\mathbf{r}_B}{r_{\mathrm{B}}^3}(d\mathbf{l}\cdot\mathbf{r}_{\mathrm{B}}) - \frac{\mathbf{r}_A}{\mathbf{r}_{\mathrm{A}}^3}(d\mathbf{l}\cdot\mathbf{r}_A)\right]. \quad (8)$$

This is in general different from zero, but it refers to a situation that is impossible to exist in nature. If A and B coincide closing the loop, $d\mathbf{F}(\mathrm{AB})\rightarrow d\mathbf{F}(\mathrm{C})=0$, as already mentioned.

The question that arises now is what happens when $d\mathbf{l}$ is on the current loop. We will postpone until later the crucial question of the divergence of the forces when considered separately and will concentrate on their difference. The situation of $d\mathbf{l}$ belonging to the loop will be arrived at gradually by a procedure to be described below. As shown in Fig. 2, $d\mathbf{l}$ is placed on the loop C and is considered to be the origin of coordinates. The force difference on $d\mathbf{l}$ due to loop C, except the part BOA, is given by Eq. (8). Using the distance s from O along the curve as variable, which is considered to increase in the direction of the current, the position vector $\mathbf{r}(s)$ of a point on C may be expanded in a series involving powers of s,

$$\mathbf{r}(s)=\mathbf{r}(0)+\mathbf{r}'(0)s+\tfrac{1}{2}\mathbf{r}''(0)s^2+\tfrac{1}{6}\mathbf{r}'''(0)s^3+\cdots, \quad (9)$$

where primes represent differentiation with respect to s. Since the origin O is on the curve, $\mathbf{r}(0)=0$. Also, for a space curve,

$$\mathbf{r}'=\mathbf{T}, \quad (10)$$

$$\mathbf{r}''=\kappa\mathbf{N}, \quad (11)$$

$$\mathbf{r}'''=-\kappa^2\mathbf{T}+\kappa'\mathbf{N}+\kappa\tau\mathbf{B}, \quad (12)$$

$$\mathbf{r}''''=-3\kappa\kappa'\mathbf{T}-(\kappa^3+\kappa\tau^2-\kappa'')\mathbf{N}+(\kappa\tau'+2\kappa'\tau)\mathbf{B}, \quad (13)$$

etc., where **T**, **N**, and **B** are the unit tangent, principal normal, and binormal vectors, respectively (Fig. 1), forming an orthonormal system, and κ is the curvature, τ is the torsion of the curve, $\kappa'=d\kappa/ds$, and $\kappa''=d^2\kappa/ds^2$. Since the expansion of $\mathbf{r}(s)$ in Eq. (9) is about the origin, the values of all the variables in Eqs. (10)–(13) are taken to be those at O when substituting in Eq. (9).

Points A and B are chosen such that $\mathbf{r}_A=\mathbf{r}(s_0)$ and $\mathbf{r}_B=\mathbf{r}(-s_0)$, i.e., they are at distance s_0 from $d\mathbf{l}$ along the curve and in opposite directions. Eventually s_0 will be allowed to tend to zero, closing the loop and incorporating $d\mathbf{l}$ to it. Since the line-current element $I_0\,d\mathbf{l}$ belongs to the loop, $I_0=I$ and $d\mathbf{l}=dl\,\mathbf{T}$. Expansion of the two terms in Eq. (8) in ascending powers of s_0 results in

$$d\mathbf{F}(\mathrm{AB})=-\frac{\mu_0 I^2}{4\pi}dl(\kappa\mathbf{N}-s_0\mathbf{c}+\cdots), \quad (14)$$

where higher powers of s_0 have been neglected and

$$\mathbf{c}=\tfrac{1}{24}[6\kappa\kappa'\mathbf{T}+(3\kappa^3-2\kappa''+2\kappa\tau^2)\mathbf{N} - 2(\kappa\tau'+2\kappa'\tau)\mathbf{B}], \quad (15)$$

with all parameters evaluated at O. As $s_0\rightarrow 0$, the loop is completed and the difference of the forces for $d\mathbf{l}$ becomes

$$d\mathbf{F}=-\frac{\mu_0 I^2}{4\pi}dl\,\kappa\mathbf{N}, \quad (16)$$

which shows that the two laws predict different forces per unit length at all points where the curvature κ is not zero.

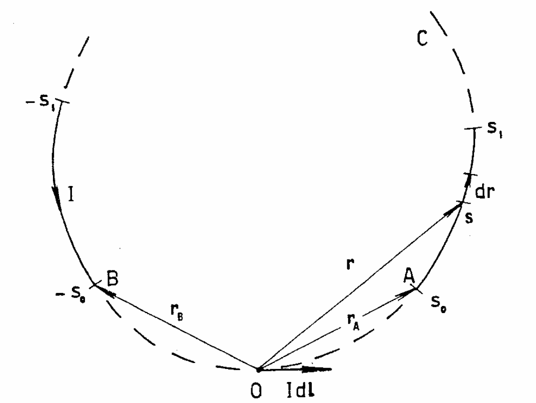

Fig. 2. The part of the current loop C that includes the line-current element $I\,d\mathbf{l}$. The ranges over which the line integrals are evaluated in examining the divergence of the Ampère and Biot–Savart forces on $d\mathbf{l}$ in Sec. IV are shown in continuous lines and correspond to values of s from $-s_1$ to $-s_0$ and from s_0 to s_1.

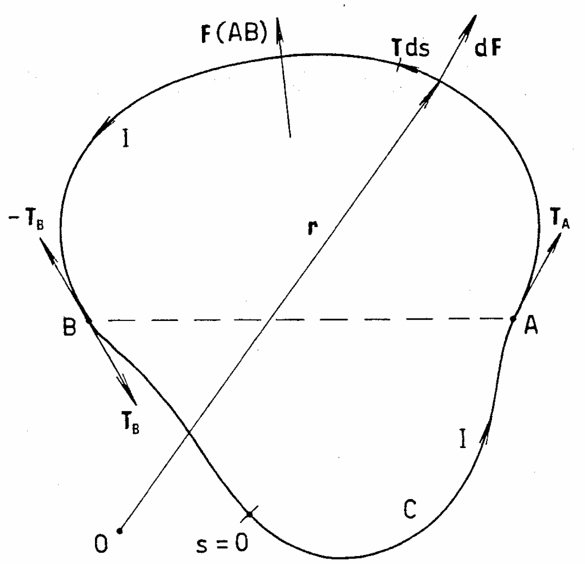

Fig. 3. The evaluation of the difference of the Ampère and Biot–Savart forces exerted by a complete current loop on a part of itself. Here, **F**(AB) is the force difference on the section from A to B (moving in the direction of the current), and $\mathbf{T}_A$ and $\mathbf{T}_B$ are the unit tangent vectors at A and B, respectively.

The difference $d\mathbf{F}$ of the forces is normal to the curve and acts along the line from the center of curvature corresponding to the position of $d\mathbf{l}$ on the loop to $d\mathbf{l}$ itself. It should be noted that even in the case of the unrealistic, infinitely thin current loop, both the Ampère and the Biot–Savart forces are normal to the curve C and, therefore, to the current element, a fact that has already been proved elsewhere.[30]

III. FORCE DIFFERENCE ON PART OF A CURRENT LOOP

Considering a current loop with current I (Fig. 3), the difference between the Ampère and Biot–Savart forces acting on an element of length ds at the position $\mathbf{r}$, is given by Eq. (16) with $dl = ds$ and $\kappa\mathbf{N} = \mathbf{r}'' = d^2\mathbf{r}/ds^2$, so that

$$d\mathbf{F} = -\frac{\mu_0 I^2}{4\pi}\frac{d^2\mathbf{r}}{ds^2}\,ds = -\frac{\mu_0 I^2}{4\pi}\,d\left(\frac{d\mathbf{r}}{ds}\right) = -\frac{\mu_0 I^2}{4\pi}\,d\mathbf{T}. \quad (17)$$

The force difference on the part of the loop from A to B (moving in the direction of I) is

$$\mathbf{F}(\mathrm{AB}) = \frac{\mu_0 I^2}{4\pi}(\mathbf{T}_\mathrm{A} - \mathbf{T}_\mathrm{B}), \quad (18)$$

based on the assumption of $\mathbf{T}(s)$ being continuous. The effects of discontinuities will be examined in Sec. IV.

Equation (18) gives the difference between the total Ampère and Biot–Savart forces exerted by the current loop C on a section of itself lying between A and B (moving in the direction of the current) as shown in Fig. 3. In general, it is different from zero, except when $\mathbf{T}_\mathrm{A} = \mathbf{T}_\mathrm{B}$ and when A and B coincide at a point where $\mathbf{T}$ is continuous. In the latter case, and since the total Ampère force is zero, it follows that according to the Biot–Savart law also, a closed current loop with continuous tangent vector $\mathbf{T}(s)$ does not exert a net force on itself and obeys Newton's third law. The difference of the forces predicted by the two laws, integrated round a closed loop without $\mathbf{T}(s)$ discontinuities, vanishes.

IV. THE EFFECTS OF DISCONTINUITIES

Some of the results derived in Secs. II and III were based on the assumption that the various integrands in the line integrals are continuous. Since the current loop C is a closed space curve, $\mathbf{r}$ is continuous. The results of Eqs. (8) and (16) are not affected by discontinuities in the derivatives of $\mathbf{r}$.

The effects of a discontinuity in $d\mathbf{r}/ds = \mathbf{T}(s)$ must be considered next. There are various ways of treating geometrically a sharp bend in the current loop. At the point of interest D (Fig. 4), the change in direction may be achieved continuously via a smooth curve EF tangential to the rest of the circuit both at E and F [Fig. 4(a)]. As E and F tend to coincide with D, with accompanying shrinking of curve EF, the force differences at D predicted by Eq. (18) are as shown in Fig. 4(b). They have a sum equal to zero and, in this respect, a discontinuity of $\mathbf{T}(s)$ at D has been avoided by the use of curve EF. If this procedure is not followed, however, the force differences corresponding to D will be as shown in Fig. 4(c), having a sum of

$$F_\mathrm{D} = (\mu_0 I^2/2\pi)\cos(\phi/2), \quad (19)$$

and direction from D along the inner bisector of the angle ϕ, which is always taken to be less than 180°. A part ADB of a closed loop containing a discontinuity of $\mathbf{T}(s)$ at D will

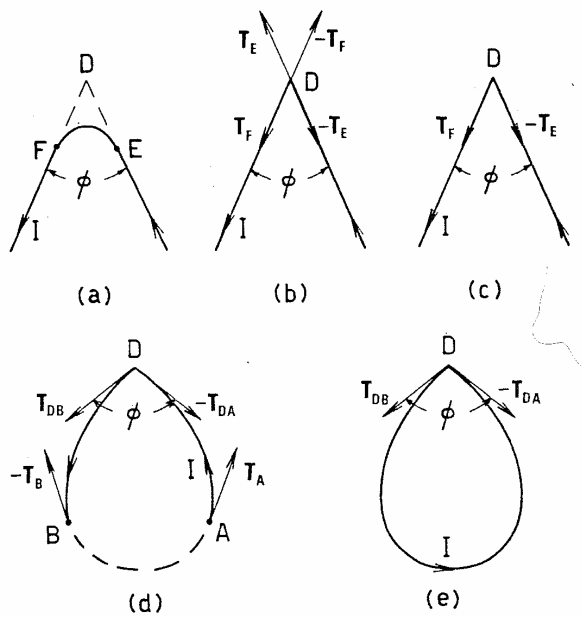

Fig. 4. The effect of a discontinuity of the tangent vector $\mathbf{T}$ at D on the force difference. Forces are found by multiplying each tangent vector $\mathbf{T}$ by $\mu_0 I^2/4\pi$.

introduce a force difference to the total, which is equal to the vector sum of two forces at D, as shown in Fig. 4(d), being tangential to the curve at either side of D and of such directions that on the side of the loop along which the current flows toward D the force difference is in the direction opposite to it, while on the other side it is in the direction of the current. For a complete loop, Fig. 4(e), the difference of Ampère and Biot–Savart forces will not be zero if there is a discontinuity in $\mathbf{T}(s)$. The presence of a discontinuity in $\mathbf{T}(s)$ will then lead to the conclusion that since the Ampère law predicts a total force on the loop equal to zero, the law of Biot–Savart predicts a nonzero total force exerted on the loop by itself, violating Newton's third law.

V. THE DIVERGENCE OF THE AMPÈRE AND BIOT–SAVART FORCES

According to Eqs. (3) and (5), the line integrals to be evaluated in obtaining the Ampère and Biot–Savart forces are

$$\int_C \frac{\mathbf{r}}{r^3}(d\mathbf{l}\cdot d\mathbf{r}), \quad (20)$$

$$\int_C \frac{d\mathbf{r}}{r^3}(d\mathbf{l}\cdot\mathbf{r}), \quad (21)$$

$$\int_C \frac{\mathbf{r}}{r^4}(d\mathbf{l}\cdot\mathbf{r})\,dr. \quad (22)$$

Proceeding as in the case of the evaluation of $d\mathbf{F}(\mathrm{AB})$, the element $d\mathbf{l}$ is placed on the curve at the origin O (Fig. 2), and $\mathbf{r}$ is expanded as in Eq. (9). The line integrals are then evaluated for $d\mathbf{l} = dl\,\mathbf{T}$ and along the segments of the curve from $-s_1$ to $-s_0$ and from s_0 to s_1. The results for the forces and their difference are

$$d\mathbf{F}_A(s_0,s_1) = -\frac{\mu_0 I^2}{4\pi} dl \left\{ \kappa\mathbf{N} \ln\left(\frac{s_1}{s_0}\right) + \frac{s_1^2 - s_0^2}{24}\left[6\kappa\kappa'\mathbf{T} + \left(\frac{7}{2}\kappa^3 - \kappa\tau^2 + \kappa''\right)\mathbf{N} + (\kappa\tau' + 2\kappa'\tau)\mathbf{B}\right] + \cdots \right\}, \quad (23)$$

$$d\mathbf{F}_{BS}(s_0,s_1) = -\frac{\mu_0 I^2}{4\pi} dl \left\{ \kappa\mathbf{N} \ln\left(\frac{s_1}{s_0}\right) + \frac{s_1^2 - s_0^2}{24}\left[\left(\frac{\kappa^3}{2} - 3\kappa\tau^2 + 3\kappa''\right)\mathbf{N} + 3(\kappa\tau' + 2\kappa'\tau)\mathbf{B}\right] + \cdots \right\}, \quad (24)$$

$$d\mathbf{F}(s_0,s_1) = d\mathbf{F}_A(s_0,s_1) - d\mathbf{F}_{BS}(s_0,s_1) = -\frac{\mu_0 I^2}{4\pi} dl (s_1^2 - s_0^2)\mathbf{c} + \cdots, \quad (25)$$

neglecting higher powers of s_0 and s_1, and with $\mathbf{c}$ as given by Eq. (15). It is seen that although the difference of the two forces converges, the Ampère and Biot–Savart forces evaluated separately, and the three integrals of Eqs. (20)–(22), all diverge (as $s_0 \to 0$) at all points where the curvature κ is not equal to zero. It might be worth mentioning here that this sort of behavior is not confined to magnetic forces alone. Diverging forces also arise in the evaluation of self-forces due to an infinitely thin curved line of charge (Coulomb forces) or of mass (gravitational forces). In these cases, the expression to be integrated along the curve is $(\mathbf{r}/r^3)ds$ and also leads to the diverging logarithmic term of Eqs. (23) and (24).

At this point, a very important feature of the mathematical analysis must be mentioned concerning the limiting procedure followed in incorporating the current element $d\mathbf{l}$ to the current loop. In Sec. II, this was achieved by having points A and B approach the origin O in a symmetrical way, i.e., with $\mathbf{r}_A = \mathbf{r}(s_0)$ and $\mathbf{r}_B = \mathbf{r}(-s_0)$. In this way, the Cauchy principal values of the integrals (20)–(22) were evaluated. There is no *a priori* reason why this procedure should be followed. In fact, by a suitable choice of asymmetric approaches of A and B to O, the integrals (20)–(22) and the tangential component of the Ampère force of Eq. (23) may be made to assume any value whatsoever. This is not the case for the Biot–Savart force of Eq. (24), which contains only the logarithmic diverging term.

In choosing to use the line-current-element forms of the laws for the evaluation of the forces of a current circuit on part of itself, one is obliged to take the Cauchy principal values of the integrals encountered, otherwise there will be indeterminacy of the tangential component of the Ampère force on a line-current element even in the simplest case of a straight conductor. Of course, the adoption of this mathematical approach removes the problem only in the case of the rectilinear parts of the loop. At points of nonzero curvature, the problem of divergence still remains and should be considered a good reason for avoiding the use of the line-current loop model in evaluating the self-forces of a circuit.

With hindsight, since the self-forces considered are finite only at points of the current loops where the curvature is zero, the analysis presented in Secs. II and III could have been greatly simplified by considering forces on rectilinear parts of the loop only. This was not done, however, and a general mathematical analysis was presented in order to pinpoint the sources of error that may exist in treatments that disregard the infinitesimal thickness of the current loop inherent in Eqs. (1) and (2), and the diverging nature of the forces involved.

VI. DISCUSSION AND CONCLUSIONS

The common features and differences of the force laws of Ampère and Biot–Savart in magnetostatics were examined, using the mathematical model of a closed space curve to represent a current loop. It is not, of course, possible to have in nature an infinitely thin current loop, either with or without discontinuities in its tangent vector. Strictly speaking, therefore, the problems examined in this analysis should never occur. Nevertheless, in the geometrical idealization of actual circuits, which is quite a common practice in solving physical problems, such situations may arise. This is the main justification for the present analysis, the conclusions of which are summarized below.

(1) The difference of the forces predicted by the two laws, as exerted on a line-current element by another such element, is an exact differential depending on the distance, relative position, and orientation of the interacting elements.

(2) Both laws give the same result for the force of a complete current loop on a line-current element, a part of a current loop, or another complete current loop, if these have no common points with the current loop exerting the force. This is true not only of the total forces but also of their distributions and the effects they produce (stresses, torques, etc.).

(3) The total Ampère force exerted by a complete current loop on a line-current element is, like the Biot–Savart force, normal to the element, both when this does and when it does not belong to the current loop exerting the force. The "longitudinal forces" sometimes attributed to the Ampère force law[12,17,18,23–27] were, therefore, shown not to exist, even when the simple model of a space curve is used to represent a current loop.

(4) For a complete current loop, the predictions of the two laws for the force per unit length acting on a line-current element belonging to the loop are the same only on rectilinear parts of the loop (where the curvature is zero), and the same is therefore true for the force distribution. Both the Ampère and the Biot–Savart forces diverge at all points of the loop with nonzero curvature. Their difference is finite, proportional to the curvature of the loop at every point, and is normal to the curve, acting along the line joining the center of curvature corresponding to a point on the loop and the point itself. This difference depends only on the local properties of the curve (namely, its curvature) at the point of interest and not on the shape or size of the current loop. When integrated around the complete loop, this difference vanishes if there are no discontinuities in the loop's tangent vector.

(5) As a consequence of (4), the two laws can be used in the evaluation of forces exerted by complete current loops on parts of themselves only when the latter are sections of straight lines and are, in addition, tangential to the rest of the loop at the point of contact. The two laws give identical results in these cases.

(6) An apparent violation of Newton's third law by the Biot–Savart self-forces of a complete current loop would

only result from arguments that disregard both the divergence of the forces considered and the fact that discontinuities in the tangent vector of a current loop cannot exist in nature.

The overall conclusion of this study is that differences of the Ampère and Biot–Savart laws in their line-current-element forms appear only when they are used in evaluating forces of a current loop on parts of itself, at points where both forces diverge, and in situations that cannot be met in nature. Some apparent differences have therefore been shown to be the result of the oversimplified geometrical model used to represent the current loop.

ACKNOWLEDGMENT

The author wishes to thank V. Peoglos of the Department of Physics, National Technical University of Athens, for many useful discussions on the subject.

[1]A. M. Ampère, *Memoires de l'Academie Royale des Sciences* (presented 1820, 1822, 1823, and 1825, issued in Paris in 1827).

[2]J. C. Maxwell, *A Treatise on Electricity and Magnetism* (Oxford U. P., London, 1873; also 3rd ed., 1904), Vol. 2, Art. 687.

[3]W. G. V. Rosser, Contemp. Phys. **3**, 28 (1961). Also in W. G. V. Rosser, *Classical Electromagnetism via Relativity* (Butterworths, London, 1968), Appendix 4.

[4]R. C. Lyness, Contemp. Phys. **3**, 453 (1961–1962).

[5]D. R. Corson and P. Lorrain, *Introduction to Electromagnetic Fields and Waves* (Freeman, San Francisco, 1962), p. 177.

[6]E. Whittaker, *A History of the Theories of Aether and Electricity* (Longmans, Green, London, 1910), pp. 88–92.

[7]C. Hering, Trans. Am. Inst. Electr. Eng. **42**, 311 (1923).

[8]F. F. Cleveland, Philos. Mag. **22**, 416 (1936).

[9]W. F. Dunton, Nature **140**, 245 (1937).

[10]S. B. L. Mathur, Philos. Mag. **32**, 171 (1941).

[11]I. A. Robertson, Philos. Mag. **36**, 32 (1945).

[12]P. Graneau, J. Appl. Phys. **53**, 6648 (1982).

[13]P. G. Moyssides and P. T. Pappas, J. Appl. Phys. **59**, 19 (1986).

[14]P. Graneau and P. N. Graneau, Nuovo Cimento **D7**, 31 (1986).

[15]P. T. Pappas, Nuovo Cimento **B76**, 189 (1983).

[16]P. Graneau, Nuovo Cimento **B78**, 213 (1983).

[17]P. T. Pappas and P. G. Moyssides, Phys. Lett. A **111**, 193 (1985).

[18]P. Graneau and P. N. Graneau, Appl. Phys. Lett. **46**, 468 (1985).

[19]V. Peoglos (to be published).

[20]D. C. Jolly, Phys. Lett. A **107**, 231 (1985).

[21]J. G. Ternan, J. Appl. Phys. **57**, 1743 (1985).

[22]C. Christodoulides, J. Phys. A: Math. Gen. **20**, 2037 (1987).

[23]P. Graneau, Phys. Lett. A **97**, 253 (1983).

[24]P. Graneau, IEEE Trans. Magn. **20**, 444 (1984).

[25]P. Graneau, J. Appl. Phys. **55**, 2598 (1984).

[26]P. Graneau, Phys. Lett. A **107**, 235 (1985).

[27]P. Graneau, J. Appl. Phys. **58**, 3638 (1985).

[28]J. G. Ternan, J. Appl. Phys. **58**, 3639 (1985).

[29]J. G. Ternan, Phys. Lett. A **115**, 230 (1986).

[30]C. Christodoulides, Phys. Lett. A **120**, 129 (1987).